SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class: Circle

12U 12M

12Z2 12Z4

11A 11B 11M1 11M2

2024

YEAR 12
TASK 4
TRIAL HSC

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- All answers, unless otherwise stated, should be left in simplified, exact form
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations
- Answers that rely totally on calculator technology may not necessarily receive full marks.

Total Marks: 100

Section I – 10 marks (pages 2 – 5)
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 38)
- Attempt all Questions in Section II
- Allow about 2 hours and 45 minutes for this section

Examiner: WB

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1 Which of the following gives the domain of $y = \dfrac{1}{\sqrt{4-5x}}$?

A. $x > \dfrac{4}{5}$

B. $x < \dfrac{4}{5}$

C. $x \geq \dfrac{4}{5}$

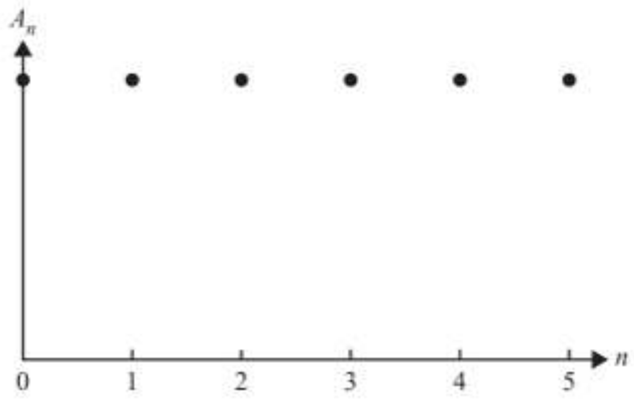
D. $x \leq \dfrac{4}{5}$

2 The average mass of 6 bags of lollies is 0.4 kg.

What could be the mass of the heaviest bag?

A. 0.3 kg B. 0.45 kg C. 2.5 kg D. 3 kg

3 The graph below represents the value A_n, in dollars, of an annuity investment for five time periods.



Which of the following recurrence relations could match this graphical representation?

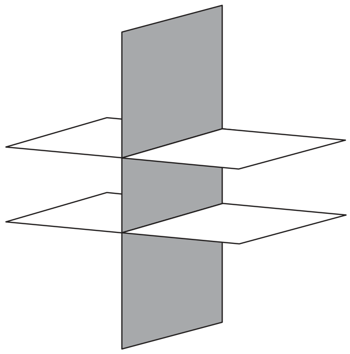
A. $A_0 = 200\,000,\ A_{n+1} = 1.015A_n - 2500$

B. $A_0 = 200\,000,\ A_{n+1} = 1.025A_n - 5000$

C. $A_0 = 200\,000,\ A_{n+1} = 1.035A_n - 5500$

D. $A_0 = 200\,000,\ A_{n+1} = 1.04A_n - 6000$

4 The geometric interpretation of a certain system of three equations with no solution is shown below.



Given two of the equations are $x + y - z = 0.5$ and $x - y - z = 0.5$, which of the following could be the third equation?

A. $2x - 2y - 2z = 1$

B. $2x + 2y - 2z = 1$

C. $2x - 2y + 2z = 3$

D. $2x + 2y - 2z = 3$

5 If $f(x) = e^{6-2x}$, what is the value of $f'(2)$?

A. e^2

B. $-e^2$

C. $2e^2$

D. $-2e^2$

6 Let $f'(x) = g'(x) + 3$, $f(0) = 1$, and $g(0) = 1$.

Which of the following is correct?

A. $f(x) = g(x) + 1$

B. $f(x) = g(x) + 3$

C. $f(x) = g(x) + 3x$

D. $f(x) = g(x) + 3x + 1$

7 The function $p(x) = x^3$ is an example of a polynomial with the property that there is a point of the graph $y = p(x)$ with zero derivative which is neither a local maximum nor a local minimum.

Which of the following polynomials has the same property?

A. $y = x^3 - 3x^2 + x$

B. $y = x^3 - 3x^2 + 2x$

C. $y = x^3 - 3x^2 + 3x$

D. $y = x^3 - 3x^2 + 4x$

8 Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m, and n are positive integers.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following MUST be true?

A. $\dfrac{m}{n}$ is an integer.

B. $\dfrac{n}{m}$ is an integer.

C. $\dfrac{a}{b}$ is an integer.

D. $\dfrac{b}{a}$ is an integer.

9 What is the maximum value of $\cos(2x + 30°)(\,1 - \cos(2x + 30°)\,)$?

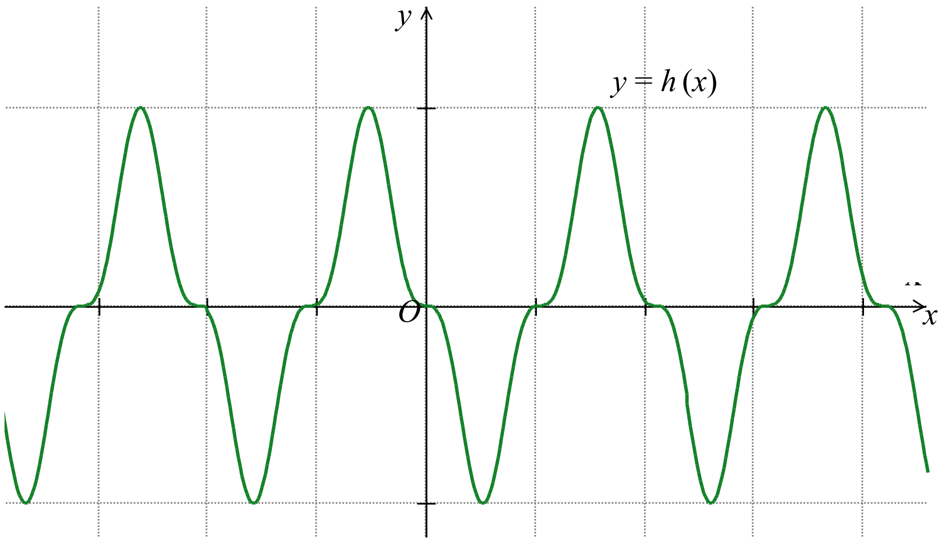
A. $\dfrac{1}{4}$

B. $\dfrac{1}{2}$

C. $\dfrac{3}{4}$

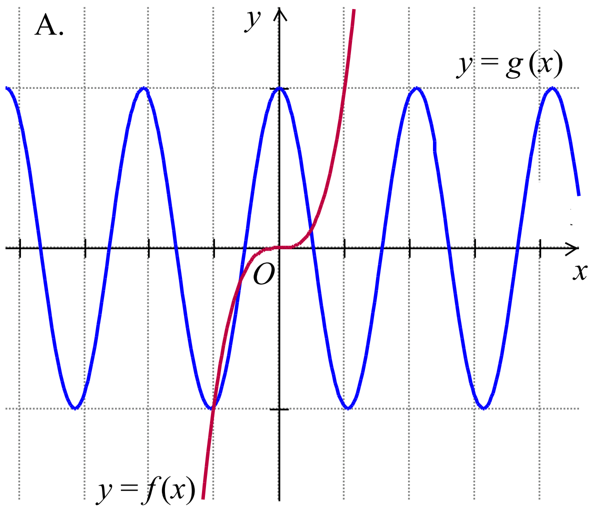
D. 1

10 The diagram shows the graph of the composite function $h(x) = f(g(x))$.

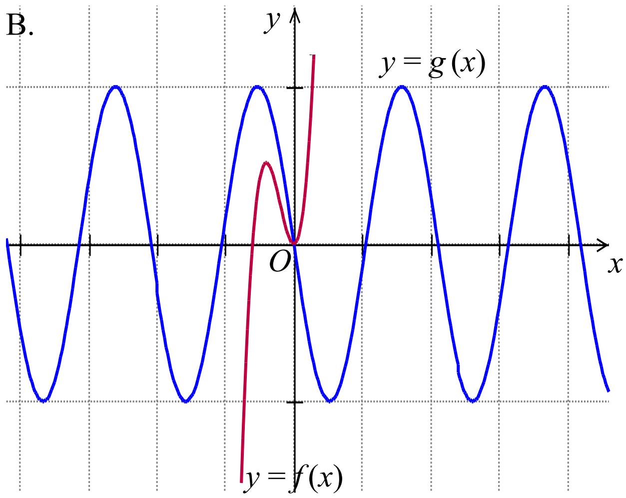


Which of the following is the best option to represent the graphs of the two functions $f(x)$ and $g(x)$?

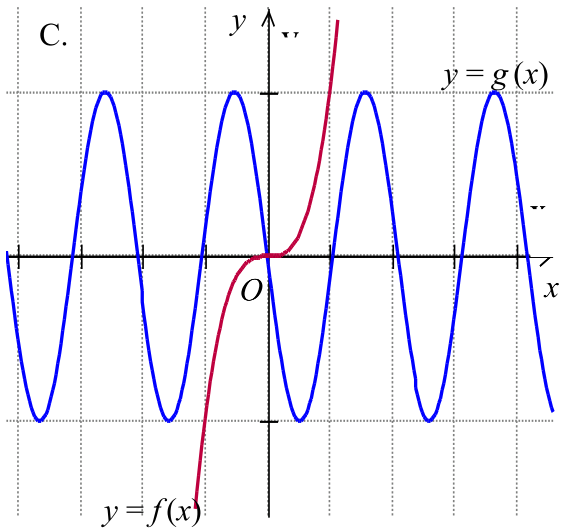
A.



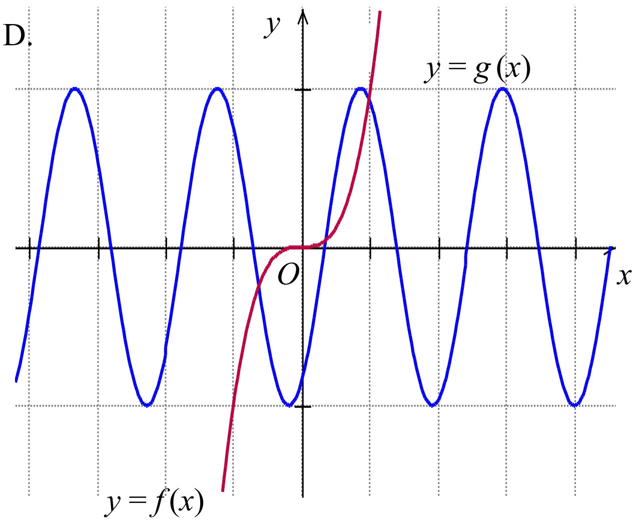
B.



C.



D.



End of Section I

NESA Number										

YEAR 12 Mathematics Advanced

HSC Task #4 (THSC)

Part A

Section II

Part A **20 marks**
Attempt Questions 11–17

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (4 marks)

Jaime owns a company producing and selling backpacks. The income function is $y = 80x$, where x is the number of backpacks sold. The cost of producing these backpacks includes a set-up cost of \$4500 and additional costs of \$30 per backpack.

(a) Write down the cost function in the form $y = mx + c$. **1**

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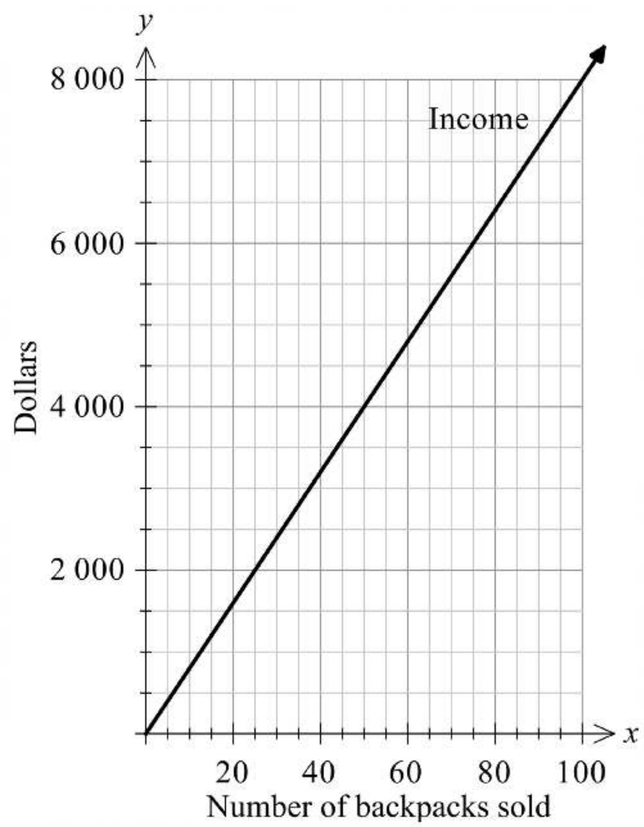
(b) Draw the cost function on the number plane below. **1**



(c) Hence, or otherwise, determine Jaime's break-even point? **2**
(i.e. how many backpacks does he need to sell to cover his costs?)

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Question 12 (2 marks)

Find $\int 2x^3 + 6x^2 - 5x \, dx$ **2**

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Question 13 (2 marks)

Find $\int \frac{2}{4x+1} \, dx$ **2**

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Question 14 (3 marks)

Evaluate $\int_0^1 4\, e^{2x} \, dx$ **3**

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Question 15 (2 marks)

Find $\int \sin 2x + \frac{1}{\sqrt{x}} \, dx$ **2**

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Question 16 (2 marks)

Find $\int \tan x \, dx$ **2**

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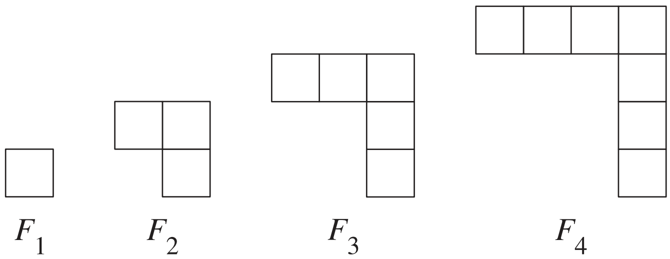
Question 17 (5 marks)

The diagram below shows the first four figures that form an arithmetic progression.



(a) Write down the constant difference. **1**

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(b) Show that $F_{88} = 175$. **2**

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(c) Show that the number of squares required to construct the first n figures is n^2. **2**

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End of Part A

Use this space to re-write any questions for Part A.

NESA Number										

YEAR 12 Mathematics Advanced

HSC Task #4 (THSC)

Part B

Section II

Part B **18 marks**
Attempt Questions 18 – 24

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 18 (3 marks)

Find the sum of the geometric progression $6 - 12 + 24 - \ldots + 1536$. **3**

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Question 19 (2 marks)

The magnitude, I, of a current varies inversely with the resistance, R, in a circuit. **2**

If $R = 10$ when $I = 24$, find the equation of I in terms of R.

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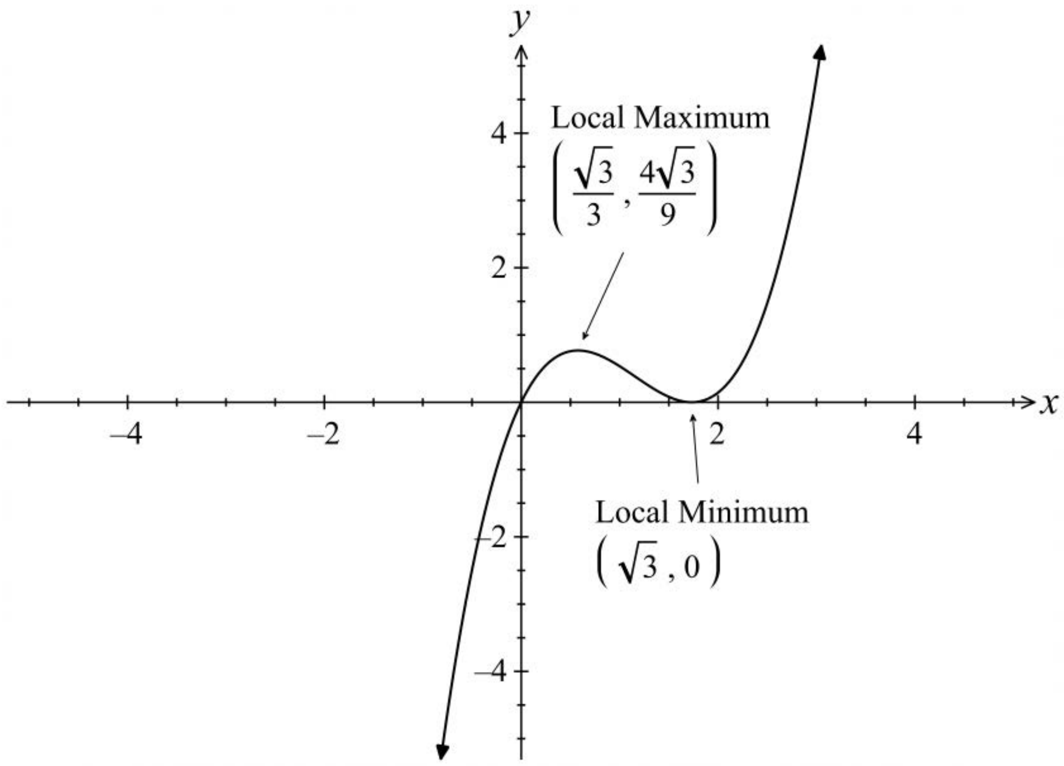
Question 20 (2 marks)

The graph below is that of $f(x) = x^3 - \left(2\sqrt{3}\right)x^2 + 3x$. **2**



The functions $g(x)$ and $h(x)$ are defined by the following equations:

$$g(x) = f(-x)$$
$$h(x) = g(x + a)$$

The graph of $h(x)$ is tangential to the x-axis at the point (5, 0).
Find the exact value of a.

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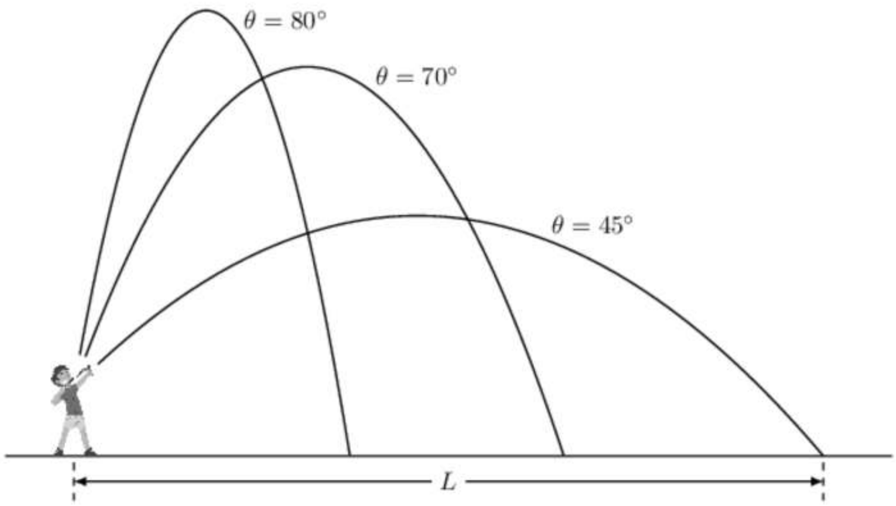
Question 21 (3 marks)

Kai launches stones from a slingshot into the air. Each time he launches a stone, he records the angle, θ, in degrees at which the stone is launched, and the horizontal distance, L, in metres, which the stone travels from his shooting position to where it first lands.

The diagram below illustrates how the stones travel in the air when launched at different angles.



Kai analyses his results and concludes:

$$\frac{dL}{d\theta} = -0.08\theta + 3.3\,,\ 0° \le \theta \le 90°.$$

(a) Determine whether the graph of L versus θ is increasing or decreasing at $\theta = 50°$ **1**

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Kai observes that when the angle is 30°, the sling stone will travel a horizontal distance of 90 m.

(b) Find an expression for L in terms of θ. **2**

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Question 22 (2 marks)

Let $f(x)$ be a function such that $\int_0^3 f(x)\,dx = 8$.

(a) Find the value of $\int_0^3 2f(x)\,dx$. **1**

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(b) If $\int_c^d f(x-2)\,dx = 8$, find the values of c and d. **1**

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Question 23 (2 marks)

Differentiate $5x^2 \tan\left(\frac{1}{x}\right)$. **2**

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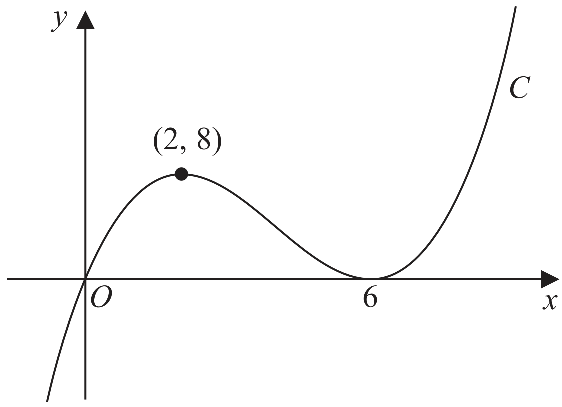
Question 24 (4 marks)



The figure above shows a sketch of a curve C with equation $y = f(x)$, where $f(x)$ is a cubic expression in x.

The curve

- passes through the origin
- has a maximum turning point at $(2, 8)$
- has a minimum turning point at $(6, 0)$

(a) Write down the values of x for which $f'(x) < 0$. **1**

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The line with equation $y = k$, where k is a constant, intersects C at only one point.

(b) Find the set of values of k. **1**

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(c) Find the equation of C. **2**

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End of Part B

Use this space to re-write any questions for Part B.

NESA Number										

YEAR 12 Mathematics Advanced

HSC Task #4 (THSC)

Part C

Section II

Part C **18 marks**
Attempt Questions 25 – 29

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 25 (3 marks)

The table below shows the future value interest factors for an annuity of $1. **3**
The contributions are made at the beginning of each year.

Period	*Interest rate per period*				
	1%	2%	3%	4%	5%
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.1010	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019

Johnny deposits $5000 into a savings account at the beginning of each year for 8 years.

For the first 6 years, the interest rate is 3% p.a., compounding annually.

During the 7^{th} and 8^{th} years, the interest rate is 2.5% p.a., compounding annually.

What is the total amount that has been added to Johnny's account at the end of 8 years?

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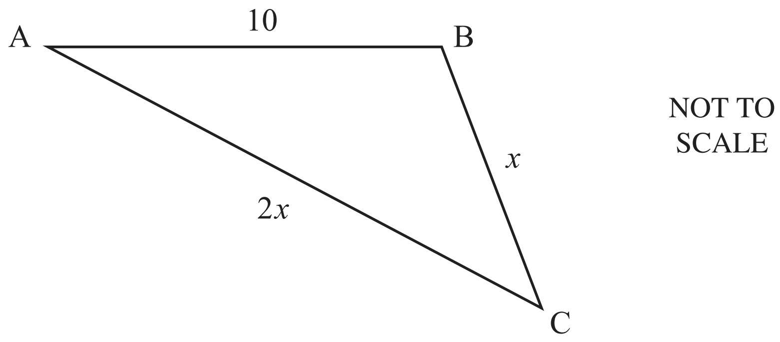
Question 26 (4 marks)

The diagram below shows triangle ABC, with $AB = 10$, $BC = x$, $AC = 2x$, and $\cos \hat{C} = \dfrac{3}{4}$. **4**



Find the area of the triangle.

Leave your answer in the form $\dfrac{p\sqrt{q}}{2}$, where p and q are integers.

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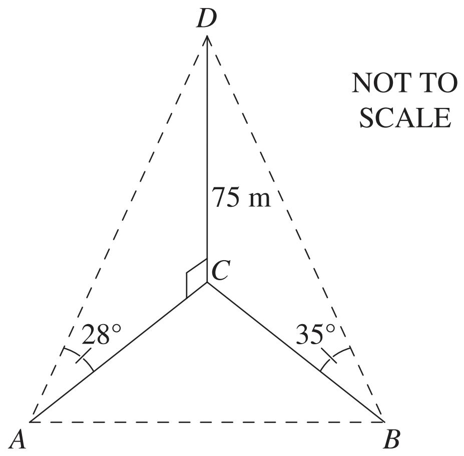
Question 27 (3 marks)

The diagram shows a 75 m vertical tower, represented by line *DC*. **3**

Points *A* and *B* are in the same horizontal plane as the base of the tower, point *C*.

The angle of elevation from point *A* to point *D* is 28°, and the angle of elevation from point *B* to point *D* is 35°.



The bearing of point *C* from point *A* is 050° T, and the bearing of point *C* from point *B* is 300° T.

Find the distance between points *A* and *B*, correct to the nearest metre.

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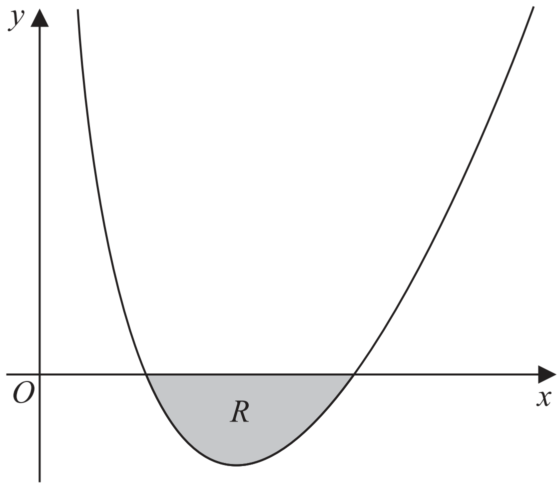
Question 28 (4 marks)

The diagram below shows a sketch of part of a curve with equation **4**

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}}, x > 0$$



The region R, shown in the diagram above, is bounded by the curve and the x-axis.

Find the exact area of R, writing your answer in the form $\frac{a\sqrt{2}+b}{5}$, where a and b are integers.

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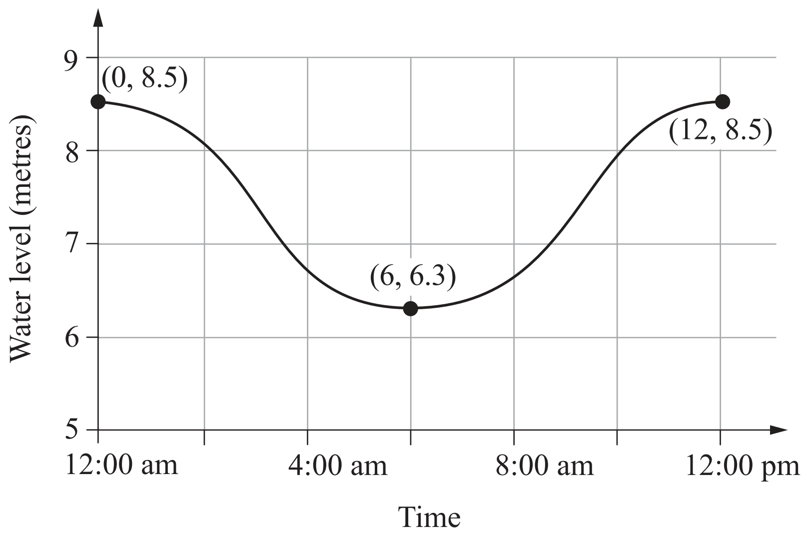
Question 29 (4 marks)

The graph below shows the water level under a bridge over a 12-hour period.



(a) Let $W(t)$ be the function that models the water level at a time t after 12. 00 am. **1**

Find the exact value of n such that $W(t) = 1.1\cos(nt) + 7.4$.

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Question 29 continues on page 24

Question 29 (continued)

(b) By sketching a graph of the derivative of the function in (a), or otherwise, find how long in the 12-hour period is the rate of change of water level more than 0.55 metres per hour. Leave your answer in hours to the nearest minute. **3**

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End of Part C

Use this space to re-write any questions for Part C.

NESA Number											

YEAR 12 Mathematics Advanced

HSC Task #4 (THSC)

Part D

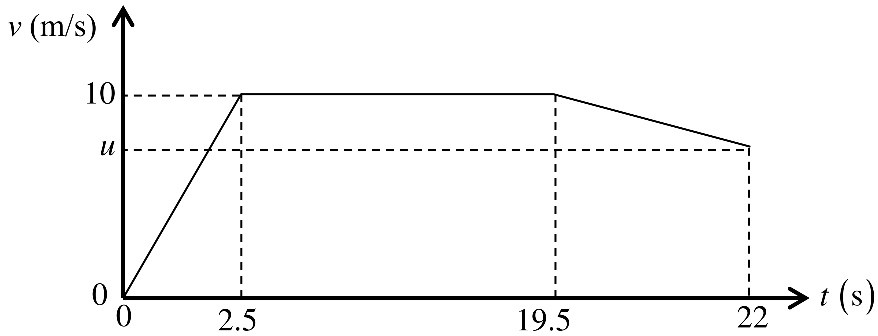
Section II

Part D **17 marks**
Attempt Questions 30 – 34

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 30 (5 marks)



The diagram above shows the speed-time graph of a sprinter running a 200 m race in a straight horizontal track.

The sprinter starts from rest and achieves, with a uniform acceleration, his top speed of 10 m/s in 2.5 s. He maintains this speed for 17 s when he experiences a cramp.

The sprinter then slows down, with a uniform acceleration. He crosses the finishing line with a speed of u m/s, 22 s after the start of the race.

By <u>only</u> using the graph above,

(a) calculate the distance covered by the sprinter until he experienced a cramp. **1**

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Question 30 continues on page 28

Question 30 (continued)

(b) show that $u = 4$ m/s **3**

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(c) calculate the acceleration of the sprinter at the last part of the race i.e. $19.5 \le t \le 22$. **1**

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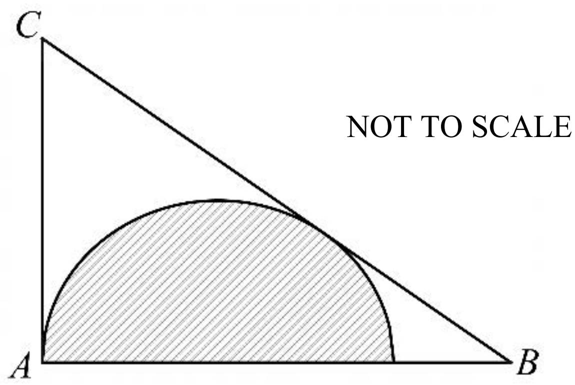
End of Question 30

Question 31 (3 marks)

In the diagram below, a semi-circle is inscribed in $\Delta\, ABC$, where $AC = 3$ cm, $AB = 4$ cm, and $BC = 5$ cm. Side BC is tangential to the semi-circle. **3**

Assuming that a radius of a circle is perpendicular to a tangent at its point of contact, find the exact area of the semi-circle.



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Question 32 (2 marks)

At a certain location, a biologist measures the width of a river to be 12 m. **2**
He also records the depth of the river at 2 m interval widths as shown.

Width (m)	0	2	4	6	8	10	12
Depth (m)	0.52	2.15	3.70	4.27	3.32	1.28	0.59

Use the trapezoidal rule to calculate an approximation to the cross-sectional area of the river.

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Question 33 (4 marks)

The function f is defined by

$$f(x) = \frac{e^{3x}}{4x^2 + k},$$

where k is a positive constant.

(a) Find a function $g(x)$ such that $f'(x) = \left(12x^2 - 8x + 3k\right) g(x)$. **2**

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(b) Given that the curve with equation $y = f(x)$ has at least one stationary point, find the range of possible values of k. **2**

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Question 34 (3 marks)

In a game, there are three different spinning wheels labelled A, B, and C.

Each of the three wheels is divided into small sectors, each of equal size.

In each sector a player can get either "Win Holiday "or "Win car".

The information about the spinning wheels is as shown in the table.

Daniel is to play the game and he is to spin a wheel at random.

	Spinning Wheels		
	A	B	C
Win Holiday	6	2	8
Win Car	3	6	2
Total	9	8	10

(a) What is the probability that Daniel selects wheel A and Win Car? **1**

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(b) Given that Daniel wins a car, what is the probability that he selected wheel C? **2**

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End of Part D

Use this space to re-write any questions for Part D.

NESA Number									

YEAR 12 Mathematics Advanced

HSC Task #4 (THSC)

Part E

Section II

Part E **17 marks**
Attempt Questions 35 – 36

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 35 (5 marks)

(a) Prove that $(1-\sin x)(\sec x+\tan x)\equiv\cos x$. **2**

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(b) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\sin^2 x(1-\sin x)(\sec x+\tan x)\,dx$. **3**

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Question 36 (7 marks)

Georgie has borrowed \$25 000 from a bank with interest accrued and repayments made monthly. The following table lists the progress for the first 6 months of the loan.

Month Number	Opening Balance	Monthly interest	Monthly Repayment	End Balance
1	\$25,000.00	\$112.50	\$400.00	\$24,712.50
2	\$24,712.50	\$111.21	\$400.00	\$24,423.71
3	\$24,423.71	\$109.91	\$400.00	\$24,133.61
4	\$24,133.61	\$108.60	\$400.00	\$23,842.21
5	\$23,842.21	\$107.29	\$400.00	\$23,549.50
6	\$23,549.50	\$105.97	\$400.00	\$23,255.48
7		**A**		**B**

(a) What is the per annum rate of interest? **1**

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(b) Determine the values of *A* and *B*. **2**

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Question 35 continues on page 36

Question 36 (continued)

(c) After how many months will Georgie have paid off the loan? **4**

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End of Question 36

Question 37 (5 marks)

(a) If $f(x)=\dfrac{\ln x}{x}$, by differentiating show that $f'(x)<0$ for $x>e$. **2**

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(b) Hence, if $e<a<b$, show that $a^b>b^a$. **3**

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End of paper

Use this space to re-write any questions for Part E.

SYDNEY BOYS HIGH SCHOOL

2024

YEAR 12

HSC TASK 4

Mathematics Advanced

Sample Solutions

NOTE: This process of checking your mark is about reading the solutions and the comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
B	B	B	D	D	C	C	D	A	C

Section II

Part A **20 marks**
Attempt Questions 11–17

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (4 marks)

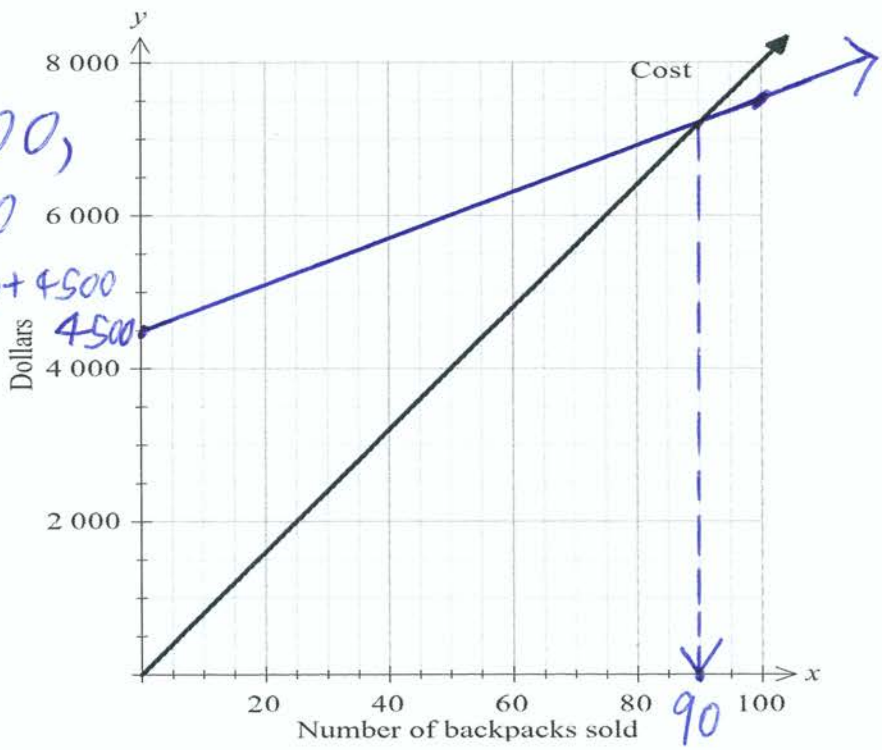
Jaime owns a company producing and selling backpacks. The income function is $y = 80x$, where x is the number of backpacks sold. The cost of producing these backpacks includes a set-up cost of \$4500 and additional costs of \$30 per backpack.

(a) Write down the cost function in the form $y = mx + c$. **1**

$y = 30x + 4500$ (done well

(b) Draw the cost function on the number plane below. **1**

When $x = 100$,
$y = 7500$
$= 30 \times 100 + 4500$
$(100, 7500)$



(a couple students didn't attempt)

(c) Hence, or otherwise, determine Jaime's break-even point? **2**
(i.e. how many backpacks does he need to sell to cover his costs?)

from graph above, no. backpacks = 90.
OR using algebra: $80x = 30x + 4500$ $(I = C)$
$50x = 4500$
$x = 90$ (done well)

Question 12 (2 marks)

Find $\int 2x^3 + 6x^2 - 5x \, dx$ **2**

$= \frac{2x^4}{4} + \frac{6x^3}{3} - \frac{5x^2}{2} + C = \frac{x^4}{2} + 2x^3 - \frac{5}{2}x^2 + C$

(half mark penalty for no +C)

Question 13 (2 marks)

Find $\int \frac{2}{4x+1} \, dx$ **2**

$= 2\int \frac{1}{4x+1} dx$

$= \frac{2}{4}\int \frac{4}{4x+1} dx = \frac{1}{2}\ln|4x+1| + C$

(half mark penalty for no abs val)

Question 14 (3 marks)

Evaluate $\int_0^1 4\, e^{2x} \, dx$ **3**

(no $\frac{1}{2}$ marks)

$= 4\left[\frac{e^{2x}}{2}\right]_0^1$

$= 2(e^2 - e^0)$

$= 2(e^2 - 1)$ OR $2e^2 - 2$

Question 15 (2 marks)

Find $\int \sin 2x + \frac{1}{\sqrt{x}} \, dx = \int \sin 2x + x^{-\frac{1}{2}} dx$ **2**

$= \frac{-\cos 2x}{2} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$= -\frac{\cos 2x}{2} + 2x^{\frac{1}{2}} + C$ or $2\sqrt{x}$

(mistakes with integrating $x^{-\frac{1}{2}}$ common)

Question 16 (2 marks)

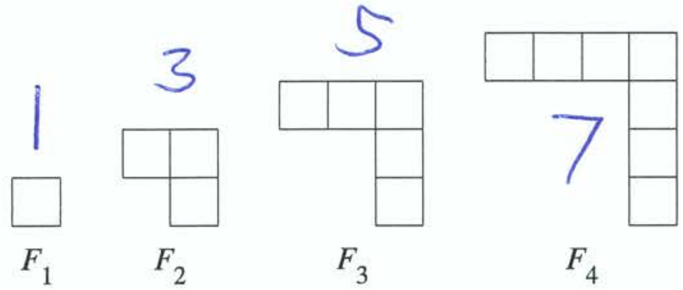
Find $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$ OR $\int \tan x \frac{\sec x}{\sec x} dx$ **2**

$= -\int \frac{-\sin x}{\cos x} dx$ $= \ln|\sec x| + C$

$= -\ln|\cos x| + C$ (These are the same)

Question 17 (5 marks)

The diagram below shows the first four figures that form an arithmetic progression.



(a) Write down the constant difference. **1**

$d = F_4 - F_3 = F_3 - F_2 = F_2 - F_1$ (only 1 needed)

$= 7 - 5 = 5 - 3 = 3 - 1$

$= 2$

(b) Show that $F_{88} = 175$. **2**

$F_n = a + (n-1)d$ (well done)

$= 1 + 2(n-1)$

$= 1 + 2n - 2$

$= 2n - 1$

$\therefore F_{88} = 2 \times 88 - 1$

$= 175$

(c) Show that the number of squares required to construct the first n figures is n^2. **2**

$S_n = [2a + (n-1)d] \times \frac{n}{2}$ where $a = 1$, $d = 2$. (using $n = 1, 2, 3$ as case studies is not acceptable for maths proofs)

$= [2 + (n-1) \times 2] \times \frac{n}{2}$

$= (2 + 2n - 2) \times \frac{n}{2}$

$= 2n \times \frac{n}{2}$

$= n^2$

End of Part A

Question 18 (3 marks)

Find the sum of the geometric progression 6 – 12 + 24 – … + 1536. **3**

$a = 6$

$r = \frac{-12}{6} = -2$

$(-2)^{n-1} = \frac{1536}{6} = 256$

$\therefore \; n = 9$

$$S_9 = \frac{a\left(1-r^9\right)}{1-r} = \frac{6\times\left(1-(-2)^9\right)}{1-(-2)} = 1026$$

Well done.

Some students were not able to find the correct value of term.

Common mistakes: 7 or 8.

Question 19 (2 marks)

The magnitude, I, of a current varies inversely with the resistance, R, in a circuit. **2**

If R = 10 when I = 24, find the equation of I in terms of R.

Let $I = \frac{k}{R}$

Sub R = 10 and I = 24 in, k =240.

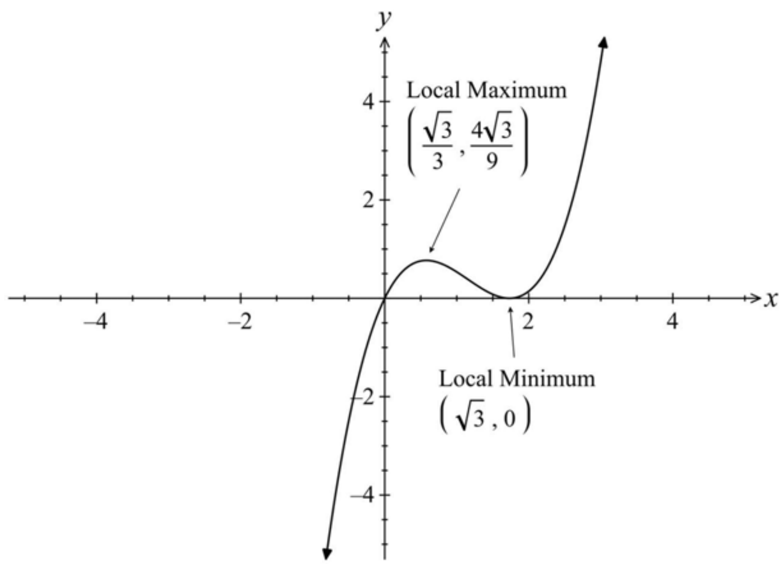
$\therefore \; I = \frac{240}{R}$

Generally well done. No half marks

Common mistake: $I = 2.4R$

Question 20 (2 marks)

The graph below is that of $f(x) = x^3 - (2\sqrt{3})x^2 + 3x$. **2**



The functions $g(x)$ and $h(x)$ are defined by the following equations:

$$g(x) = f(-x)$$
$$h(x) = g(x + a)$$

The graph of $h(x)$ is tangential to the x-axis at the point $(5, 0)$.
Find the exact value of a.

Method 1:

$f(-x)$ reflects in y-axis to obtain $g(x)$

$\therefore$ the graph of $g(x)$ is tangential to the x-axis at the point $\left(-\sqrt{3},\ 0\right)$.

$g(x)$ shifts a units left (i.e. $-a$ units right) to obtain $h(x)$ which is tangential to the x-axis at the point $(5, 0)$

$\therefore -a = 5 + \sqrt{3}$

$\therefore a = -5 - \sqrt{3}$

Continued over the page

Question 20 (continued)

Method 2:

$h(x) = f(-(x+a))$

$\therefore\ 5 = -(\sqrt{3} + a)$

$\therefore\ a = -5 - \sqrt{3}$

Poorly done.

This question requires students to demonstrate their understanding of graph transformations. Therefore, no mark was awarded if students only got the equation of $h(x)$.

Many students tried to use a very complicated method to answer the question but they could not complete this satisfactorily to get full marks. They

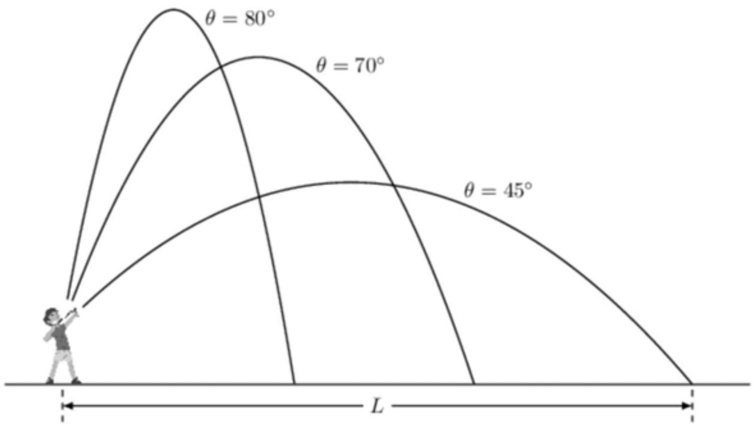
- *made careless calculation mistakes*
- *were struggling in solving quadratic equation for* a
- *had two values or did not justify why* $a \neq -5 - \dfrac{\sqrt{3}}{3}$

Common mistakes: $-5+\sqrt{3}$, $5-\sqrt{3}$, $5+\sqrt{3}$

Question 21 (3 marks)

Kai launches stones from a slingshot into the air. Each time he launches a stone, he records the angle, θ, in degrees at which the stone is launched, and the horizontal distance, L, in metres, which the stone travels from his shooting position to where it first lands.

The diagram below illustrates how the stones travel in the air when launched at different angles.



Kai analyses his results and concludes:

$$\frac{dL}{d\theta} = -0.08\theta + 3.3\,,\ 0° \le \theta \le 90°.$$

(a) Determine whether the graph of L versus θ is increasing or decreasing at $\theta = 50°$ **1**

$\frac{dL}{d\theta} = -0.08 \times 50 + 3.3 = -0.7 < 0$

$\therefore$ It is decreasing.

Well done.

Half marks were deducted if students did not justify their answer.
Few calculation mistakes.

Kai observes that when the angle is 30°, the sling stone will travel a horizontal distance of 90 m.

(b) Find an expression for L in terms of θ. **2**

$L = -0.04\theta^2 + 3.3\theta + C$

$\therefore\ 90 = -0.04 \times 30^2 + 3.3 \times 30 + C$

$\therefore\ C = 27$

$\therefore\ L = -0.04\theta^2 + 3.3\theta + 27$

Well done.
Common mistakes:

- *converted degrees to radians*
- *calculation mistake, incorrect value of C*
- *did not simplify their answers*

Question 22 (2 marks)

Let $f(x)$ be a function such that $\int_0^3 f(x)\,dx = 8$.

(a) Find the value of $\int_0^3 2f(x)\,dx$. **1**

$\int_0^3 2f(x)dx = 2\int_0^3 f(x)dx = 16$

Well done. No half marks.

Common mistake: 4

(b) If $\int_c^d f(x-2)\,dx = 8$, find the values of c and d. **1**

$f(x)$ shifts 2 units right to gain $f(x-2)$,

$\therefore\ d = 3 + 2 = 5$

$c = 0 + 2 = 2$

Poorly done. No half marks.

Some students did not attempt this question.
Common mistake: $c = -2, d = 1$

Question 23 (2 marks)

Differentiate $5x^2 \tan\left(\frac{1}{x}\right)$. **2**

$$\frac{d}{dx}\left(5x^2 \tan\left(\frac{1}{x}\right)\right) = 10x\tan\left(\frac{1}{x}\right) + 5x^2 \times \left(-x^{-2}\right)\sec^2\left(\frac{1}{x}\right)$$

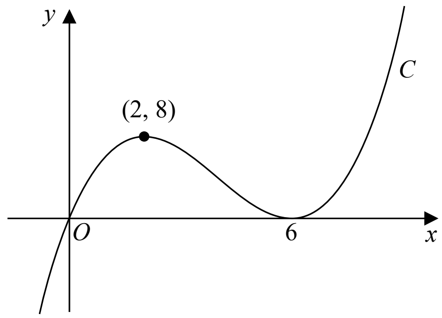
$$= 10x\tan\left(\frac{1}{x}\right) - 5\sec^2\left(\frac{1}{x}\right)$$

Generally well done.

All students were able to correctly differentiate $5x^2$*, but some were struggling in differentiating* $\tan\frac{1}{x}$.

Most students were able to correctly apply the product rule, but some were not able to correctly simplify their answers.

Question 24 (4 marks)



The figure above shows a sketch of a curve C with equation $y = f(x)$, where $f(x)$ is a cubic expression in x.

The curve

- passes through the origin
- has a maximum turning point at $(2, 8)$
- has a minimum turning point at $(6, 0)$

(a) Write down the values of x for which $f'(x) < 0$. **1**

$(2, 6)$ or $2 < x < 6$

Well done.

Few students included 2 and 6, or listed all possible integers.

The line with equation $y = k$, where k is a constant, intersects C at only one point.

(b) Find the set of values of k. **1**

$(-\infty, 0) \cup (8, \infty)$ or $x < 0, x > 8$

Poorly done.

Some students only had either $(-\infty, 0)$ or $(8, \infty)$ as their answers, but not both
Many students misread the graph and had $(-\infty, 0) \cup (2, \infty)$ as their answer.

(c) Find the equation of C. **2**

The graph passes through the origin, has single root at $x = 0$ and double roots at $x = 6$

$\therefore C = ax(x-6)^2$

Sub $(2, 8)$ in, $8 = a \times 2(2-6)^2$

$$\therefore\ a = \frac{1}{4}$$

$$\therefore\ C = \frac{1}{4}x(x-6)^2$$

Poorly done.
Some students did not attempt this question.
Common errors:

- *forgot the coefficient of $x(x-6)^2$*
- *found the equation of C'*
- *calculation mistakes, including incorrectly solved simultaneous equations.*

End of Part B Solutions

Question 25 (3 marks)

The table below shows the future value interest factors for an annuity of $1. **3**
The contributions are made at the beginning of each year.

Period	*Interest rate per period*				
	1%	2%	3%	4%	5%
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.1010	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019

Johnny deposits $5000 into a savings account at the beginning of each year for 8 years.

For the first 6 years, the interest rate is 3% p.a., compounding annually.

During the 7th and 8th years, the interest rate is 2.5% p.a., compounding annually.

What is the total amount that has been added to Johnny's account at the end of 8 years?

First 6 years: $5000 \times 6.4684 = \$32342$

$A_7 = (5000 + 32342) \times 1.025 = \$43275.55$

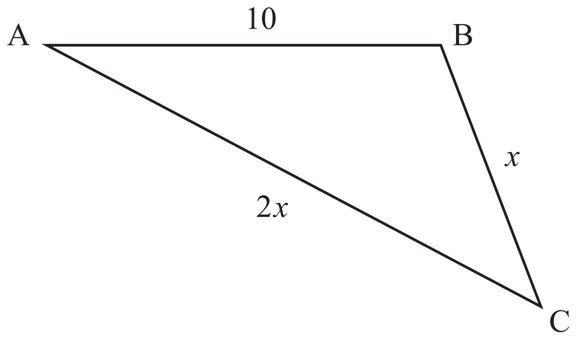
$A_8 = (5000 + A_7) \times 1.025 = \$44357.44$

$\therefore \$44357.44$

Poorly done, many students didn't realise $5000 is deposited at the beginning of each year.

Question 26 (4 marks)

The diagram below shows triangle ABC, with $AB = 10$, $BC = x$, $AC = 2x$, and $\cos \hat{C} = \frac{3}{4}$. **4**



NOT TO SCALE

Find the area of the triangle.

Leave your answer in the form $\frac{p\sqrt{q}}{2}$, where p and q are integers.

$$\cos C = \frac{(2x)^2 + x^2 - 10^2}{2(2x)(x)} = \frac{3}{4}$$

$$\frac{5x^2 - 100}{4x^2} = \frac{3}{4}$$

$$5x^2 - 100 = 3x^2$$

$$2x^2 = 100$$

$$x = \sqrt{50} \quad (x > 0)$$

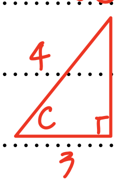
$$A = \frac{1}{2}(2x)(x)\sin C$$

$$= x^2 \sin C$$

$$= \frac{50\sqrt{7}}{4}$$

$$= \frac{25\sqrt{7}}{2}\ u^2$$

$\because \cos C = \frac{3}{4}$



$\sqrt{4^2 - 3^2} = \sqrt{7}$

$\sin C = \frac{\sqrt{7}}{4}$

Generally well done.

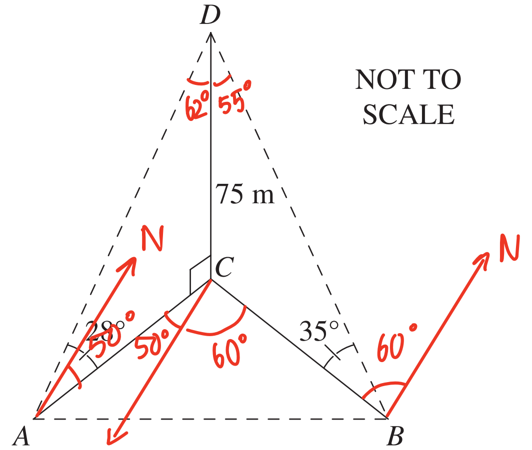
Few calculation errors, e.g. $\sqrt{4^2 - 3^2} = 5$

Question 27 (3 marks)

The diagram shows a 75 m vertical tower, represented by line *DC*. **3**

Points *A* and *B* are in the same horizontal plane as the base of the tower, point *C*.

The angle of elevation from point *A* to point *D* is 28°, and the angle of elevation from point *B* to point *D* is 35°.



The bearing of point *C* from point *A* is 050° T, and the bearing of point *C* from point *B* is 300° T.

Find the distance between points *A* and *B*, correct to the nearest metre.

$\angle ACB = 50° + 60° = 110°$

$\tan 62° = \frac{AC}{75}$

$AC = 75\tan 62°$

Similarly, $BC = 75\tan 55°$

$AB = \sqrt{(AC)^2 + (BC)^2 - 2(AC)(BC)\cos 110°}$

$= \sqrt{(75\tan 62°)^2 + (75\tan 55°)^2 - 2(75\tan 62°)(75\tan 55°)\cos 110°}$

$= 204$ m (nearest m)

Question 27 (3 marks)

The diagram shows a 75 m vertical tower, represented by line *DC*. Points *A* and *B* are in the same horizontal plane as the base of the tower, point *C*, ~~and point *A* is west of point *B*~~. **3**

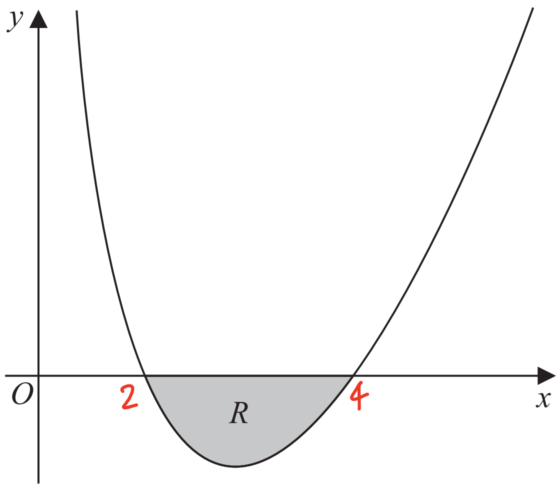
The angle of elevation from point *A* to point *D* is 28°, and the angle of elevation from point *B* to point *D* is 35°.

- Many students didn't notice the line through the question. This means ignore that part of the question.
- Few calculation errors, e.g. using sin or cos instead of tan.

Question 28 (4 marks)

The diagram below shows a sketch of part of a curve with equation **4**

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}}, x > 0$$



The region R, shown in the diagram above, is bounded by the curve and the x-axis.

Find the exact area of R, writing your answer in the form $\dfrac{a\sqrt{2}+b}{5}$, where a and b are integers.

$y = \frac{1}{4}x^{-\frac{1}{2}}(x^2 - 6x + 8)$

$= \frac{1}{4}(x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 8x^{-\frac{1}{2}})$

$R = \left| \int_2^4 \frac{1}{4}(x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 8x^{-\frac{1}{2}})\, dx \right|$

$= \left| \frac{1}{4}\left[\frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 16x^{\frac{1}{2}}\right]_2^4 \right|$

$= \frac{1}{4}\left| \left[\frac{2}{5}x^2\sqrt{x} - 4x\sqrt{x} + 16\sqrt{x}\right]_2^4 \right|$

$= \frac{1}{4}\left| \left[\frac{64}{5} - 32 + 32 - \left(\frac{2}{5}4\sqrt{2} - 4 \cdot 2\sqrt{2} + 16\sqrt{2}\right)\right]\right|$

$= \frac{1}{4}\left| \frac{64}{5} - \frac{48}{5}\sqrt{2} \right|$

$= \frac{1}{5}\left| \frac{64}{4} - \frac{48}{4}\sqrt{2} \right|$

$= \frac{1}{5}\left| 16 - 12\sqrt{2} \right|$

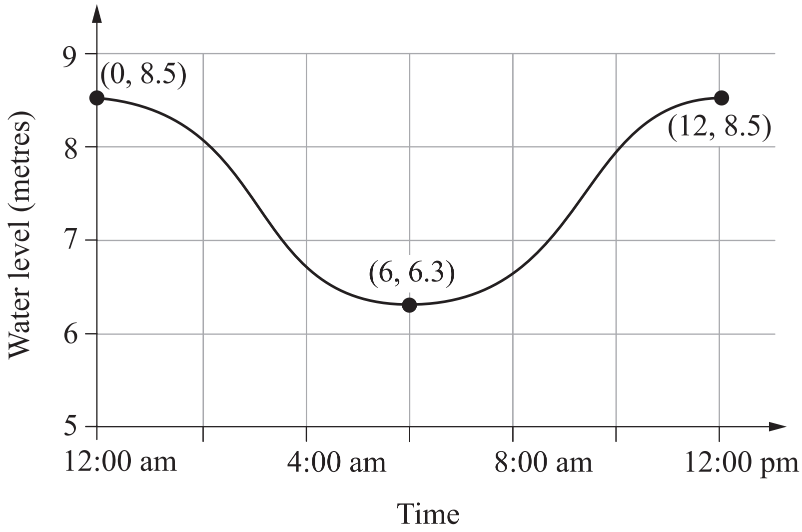
$= \dfrac{12\sqrt{2} - 16}{5} u^2$

Poorly done. Many students forgot to take the absolute value.

Students need to take care with negatives and review how to change indices to surds.

Question 29 (4 marks)

The graph below shows the water level under a bridge over a 12-hour period.



(a) Let $W(t)$ be the function that models the water level at a time t after 12. 00 am. **1**

Find the exact value of n such that $W(t) = 1.1\cos(nt) + 7.4$.

Period $= 12 = \frac{2\pi}{n}$

$n = \frac{2\pi}{12}$

$= \frac{\pi}{6}$

Some students wrote 30°. This is incorrect since 1.1 and 7.4 are lengths. Degrees cannot be mixed with lengths, n needs to be in radians.

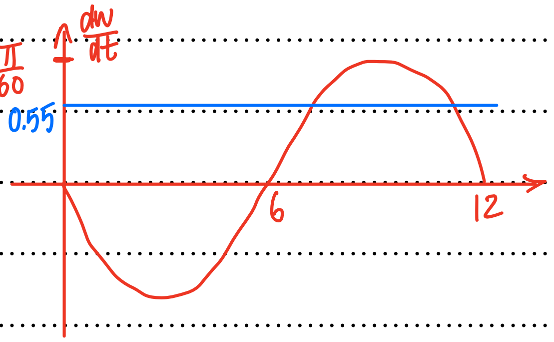
Question 29 continues on page 24

Question 29 (continued)

(b) By sketching a graph of the derivative of the function in (a), or otherwise, find how long in the 12-hour period is the rate of change of water level more than 0.55 metres per hour? Leave your answer in hours to the nearest minute. **3**

$w = 1.1\cos\frac{\pi}{6}t + 7.4$

$\frac{dw}{dt} = -\frac{11\pi}{60}\sin\frac{\pi}{6}t$



$-\frac{11\pi}{60}\sin\frac{\pi}{6}t = 0.55$

$\sin\frac{\pi}{6}t = \frac{-3}{\pi}$ Q3/Q4 (sin negative)

Related angle : $\sin^{-1}\left(\frac{3}{\pi}\right)$

$\frac{\pi}{6}t = \pi + \sin^{-1}\left(\frac{3}{\pi}\right),\ 2\pi - \sin^{-1}\left(\frac{3}{\pi}\right)$

$t = \frac{6}{\pi}\left[\pi + \sin^{-1}\left(\frac{3}{\pi}\right)\right],\ \frac{6}{\pi}\left[2\pi - \sin^{-1}\left(\frac{3}{\pi}\right)\right]$

$= 8°25'27.91'',\ 9°34'32.09''$

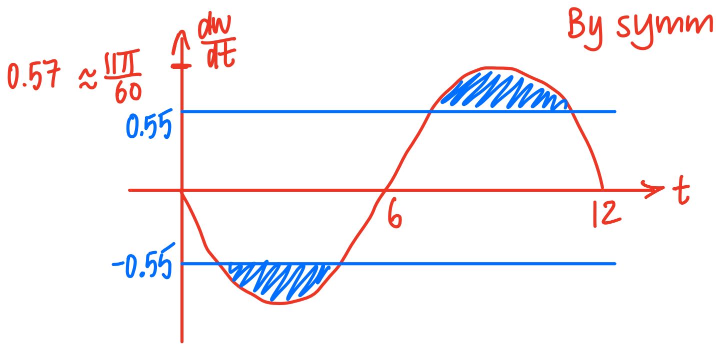
$= 8:25:30$ am , $9:34:30$ am (nearest 30 second)

$\therefore\ 9:34:30 - 8:25:30 = 1$ hr 9 min (nearest minute)

Rate of change $> 0.55 \rightarrow \pm\frac{dw}{dt} > 0.55$

$\frac{dw}{dt} > 0.55$ $\quad$ $\frac{dw}{dt} < -0.55$

By symmetry, (1hr 9min) × 2 = 2hr 18min

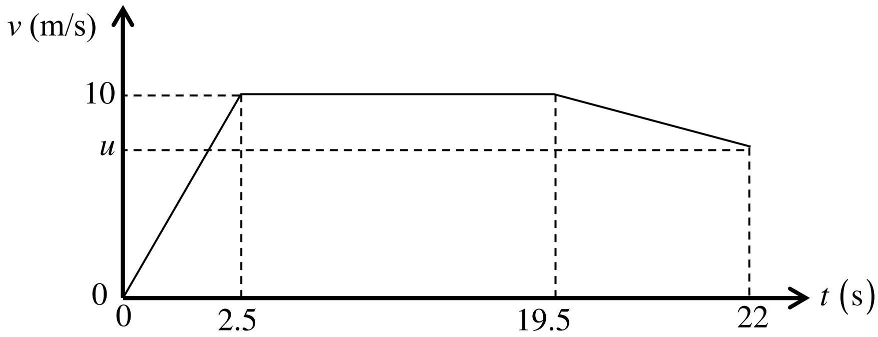


Most students found the first derivative, but struggled to solve the inequality.

A diagram would have helped, though no marks were awarded for this.

End of Part C

Question 30 (5 marks)



The diagram above shows the speed-time graph of a sprinter running a 200 m race in a straight horizontal track.

The sprinter starts from rest and achieves, with a uniform acceleration, his top speed of 10 m/s in 2.5 s. He maintains this speed for 17 s when he experiences a cramp.

The sprinter then slows down, with a uniform acceleration. He crosses the finishing line with a speed of u m/s, 22 s after the start of the race.

By <u>only</u> using the graph above,

(a) calculate the distance covered by the sprinter until he experienced a cramp. **1**

The area under a velocity-time graph is equal to the change in distance.

As such, the distance travelled by the sprinter can be determined by evaluating the area under the curve from $t=0$ to $t=19.5$.

$$\begin{aligned} d &= \frac{1}{2}(10)(2.5)+(10)(195.-2.5) \\ &= 182.5 \text{ m} \end{aligned}$$

1 mark for a valid approach that used <u>only</u> the graph.

No half marks.

Solutions using calculus or physics equations were NOT valid.

Generally done quite well.

Common errors included:

- Not reading the question properly and calculating the wrong area.
- Forgetting to include the first 2.5 seconds as part of the distance.
- Incorrectly having 17.5 – 2.5 for the rectangle length.

Question 30 (continued)

(b) show that $u = 4$ m/s. **3**

The remaining area under the curve is equal to the remaining distance in the race.

$$\frac{1}{2}(u+10)(22-19.5) = 200-182.5$$
$$\frac{1}{2}(u+10)(2.5) = 17.5$$
$$u+10 = 14$$
$$u = 4 \text{ m/s} \qquad \text{as required.}$$

1 mark for using part (a) to determine the remaining distance.

1 mark for giving an expression for the distance in terms of *u*.

1 mark for solving for *u* to show the required result.

Generally done ok.

Common errors included:

- Writing $u-10$ instead of $u+10$ for the remaining distance expression.
- Carrying over an error from part (a) and then trying to force an invalid proof.

A common alternate approach was dividing 17.5 by 2.5 to show that 7 m/s was the average speed over the last section, so the average of 10 and u must be 7, which gives the answer.

(c) calculate the acceleration of the sprinter at the last part of the race i.e. $19.5 \leq t \leq 22$. **1**

The acceleration from $t = 19.5$ to $t = 22$ is equal to the gradient of the curve over that time.

$$a = \frac{10-4}{19.5-22} = -\frac{12}{5} = -2.4 \text{ m/s}^2$$

1 mark for correctly determining the acceleration of the sprinter.

Generally done very well.

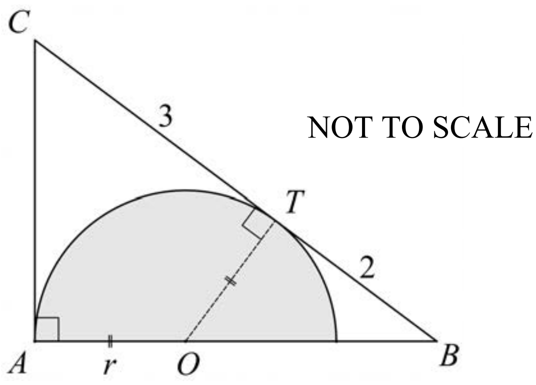
Common errors included:

- Having the incorrect sign.
- Having a change in speed of 4, instead of 6.

Question 31 (3 marks)

In the diagram below, a semi-circle is inscribed in $\Delta\, ABC$, where $AC = 3$ cm, $AB = 4$ cm, and $BC = 5$ cm. Side BC is tangential to the semi-circle. **3**

Assuming that a radius of a circle is perpendicular to a tangent at its point of contact, find the exact area of the semi-circle.



Let the centre of the semicircle be O.
Let the point of tangency on BC be T.
Let the radius of the semicircle be r.

Note that triangle ABC is right-angled since its sides satisfy Pythagoras' theorem.

$OB = 4 - r$ (given)
$CT = 3$ (intersection of tangents)
$TB = 2$ (adjacent intervals)

Considering Pythagoras' theorem in the right triangle OTB:

$$\begin{aligned} r^2 + 2^2 &= (4-r)^2 \\ r^2 + 4 &= 16 - 8r + r^2 \\ 8r &= 12 \\ r &= \frac{3}{2} \end{aligned}$$

Therefore, the area of the semicircle is $\frac{9\pi}{8}$ cm^2.

1 mark for recognising the length of *OB*.
1 mark for determining the length of *CT*.
1 mark for calculating the radius of the semicircle. (Area of the semicircle is assumed.)

Note: Congruent triangles OAC and OTC can also be used to find the length of CT. (RHS)

Generally done ok. Marks were generally lost from not having a valid cohesive approach.
An alarming number of students simplified $\frac{12}{8}$ to be $\frac{4}{3}$, which is mathematically untrue.

There were many ways to do this problem. See next page for alternate solutions.

Alternate method 1: Trigonometry

Using the diagram, we can recognise that $\sin \angle B = \frac{3}{5} = \frac{r}{4-r}$.

Solving this equation gives the correct radius.

Alternate method 2: Similar triangles

Using right angles and a common angle, we can show that triangles *ABC* and *TBO* are similar.

By the ratio of corresponding sides in similar triangles, we get $\frac{3}{4} = \frac{r}{2}$.

Solving this equation gives the correct radius.

Alternate method 3: Area method

The area of triangle *ABC* is 6 cm^2.

By constructing *OC*, we can divide the area into the triangles *AOC* and *COB*.

As such, we can construct the equation $\frac{3r}{2} + \frac{5r}{2} = 6$.

Solving this equation gives the correct radius.

Question 32 (2 marks)

At a certain location, a biologist measures the width of a river to be 12 m. **2**
He also records the depth of the river at 2 m interval widths as shown.

Width (m)	0	2	4	6	8	10	12
Depth (m)	0.52	2.15	3.70	4.27	3.32	1.28	0.59

Use the trapezoidal rule to calculate an approximation to the cross-sectional area of the river.

The trapezoidal rule is given on the **reference sheet** as:

$$A \approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\left[f(x_1)+\ldots+f(x_{n-1})\right]\right\}$$

where a is the lower bound and b is the upper bound.

In this case, we have that $a=0$ and $b=12$, and x_1 up to x_{n-1} are the even values between them.
Since there are 5 values between a and b, we get $n-1=5 \rightarrow n=6$.

$$\begin{aligned} A &\approx \frac{12-0}{2\times 6}\left\{f(0)+f(12)+2\left[f(2)+f(4)+f(6)+f(8)+f(10)\right]\right\} \\ &= \frac{12}{12}\left\{0.52+0.59+2\left[2.15+3.70+4.27+3.32+1.28\right]\right\} \\ &= 30.55\text{ m}^2 \end{aligned}$$

1 mark for correctly substituting all function values.
1 mark for correctly determining the value of *n*.

Note: This problem can also be solved using multiple applications of the trapezoidal rule.

Generally done well.

Common errors included:

- Having $n=7$.
- Forgetting the factor of 2 in the denominator of the outer term.

Students are also reminded to write all the function values when doing a trapezoidal rule question, using an ellipsis to truncate the solution was not accepted.

The calculator work was also very sloppy – this was not penalised but it should be noted that many evaluated answers did not match the previous line of work.

Question 33 (4 marks)

The function f is defined by

$$f(x)=\frac{e^{3x}}{4x^2+k},$$

where k is a positive constant.

(a) Find a function $g(x)$ such that $f'(x)=(12x^2-8x+3k)g(x)$. **2**

To find the derivative, we can use the quotient rule.

On the **reference sheet** the quotient rule is given by: $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

Let $u=e^{3x}$ and $v=4x^2+k$. Then $\dfrac{du}{dx}=3e^{3x}$ and $\dfrac{dv}{dx}=8x$.

$$\begin{aligned}
f'(x)&=\frac{dy}{dx}\\
&=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}\\
&=\frac{(4x^2+k)(3e^{3x})-(e^{3x})(8x)}{(4x^2+k)^2}\\
&=\frac{e^{3x}}{(4x^2+k)^2}\left[(4x^2+k)(3)-8x\right]\\
&=\frac{e^{3x}}{(4x^2+k)^2}(12x^2-8x+3k)
\end{aligned}$$

$$\therefore \qquad g(x)=\frac{e^{3x}}{(4x^2+k)^2}$$

1 mark for attempting to use the quotient rule.
1 mark for correctly differentiating and factorising the derivative to isolate $g(x)$.

Generally done quite well.

Question 33 (continued)

(b) Given that the curve with equation $y = f(x)$ has at least one stationary point, find the range of possible values of k. **2**

For there to be at least one stationary point, there must be at least one x-values that gives $f'(x) = 0$.

Since $e^{3x} > 0$ and $\left(4x^2 + k\right)^2 \geq 0$, we get that $g(x) > 0$.

This means that the requirement for at least one stationary point is that $12x^2 - 8x + 3k = 0$ has at least one solution. This means that the discriminant must be greater than or equal to zero.

$$\begin{aligned} 8^2 - 4(12)(3k) &\geq 0 \\ 64 &\geq 144k \\ k &\leq \frac{64}{144} \\ &= \frac{4}{9} \end{aligned}$$

Since k is positive, the range of possible values is $0 < k \leq \frac{4}{9}$.

1 mark for recognising that $12x^2 - 8x + 3k = 0$ is needed for at least one stationary point.
1 mark for finding the range of values for *k*.

Generally done ok.

Common errors included:

- Incorrect inequality symbol (having not inclusive, or wrong direction).
- Trying to solve the quadratic equation instead of the discriminant inequality.

Very few students remembered that k also needed to be positive. (This was not penalised.)

Question 34 (3 marks)

In a game, there are three different spinning wheels labelled *A*, *B*, and *C*.

Each of the three wheels is divided into small sectors, each of equal size.

In each sector a player can get either "Win Holiday "or "Win car".

The information about the spinning wheels is as shown in the table.

Daniel is to play the game and he is to spin a wheel at random.

	Spinning Wheels		
	A	*B*	*C*
Win Holiday	6	2	8
Win Car	3	6	2
Total	9	8	10

(a) What is the probability that Daniel selects wheel *A* and Win Car? **1**

$$\begin{aligned} P(\text{wheel A} \cap \text{Win Car}) &= P(\text{wheel A}) \times P(\text{Win Car on wheel A}) \\ &= \frac{1}{3} \times \frac{3}{9} \\ &= \frac{1}{9} \end{aligned}$$

1 mark for the correct answer.

Generally done quite well.

While the working

$$\frac{3}{9+8+10}$$

gave the correct answer, this approach is invalid as it makes the incorrect assumption that the wheels aren't distinct (rather it assumes that the three wheels are combined into one).
As such, any approach showing this approach could not be awarded marks.

(b) Given that Daniel wins a car, what is the probability that he selected wheel *C*? **2**

$$\begin{aligned}
P(\text{wheel C}|\text{Win Car}) &= \frac{P(\text{wheel C } \cap \text{ Win Car})}{P(\text{Win Car})} \\
&= \frac{\frac{1}{3}\times\frac{2}{10}}{\frac{1}{3}\times\frac{3}{9}+\frac{1}{3}\times\frac{6}{8}+\frac{1}{3}\times\frac{2}{10}} \\
&= \frac{12}{77}
\end{aligned}$$

1 mark for giving the definition form for conditional probability.
1 mark for the correct answer.

Generally done poorly.

Common errors included:

- Writing the numerator as $P(\text{wheel C})\times P(\text{Win Car})$ incorrectly assumes that the two events are independent, which they are not.
- The answer $\frac{2}{11}$ makes the same error of assuming that all wheels are together.

End of Part D Solutions

## Section II		Part E Solutions		17 Marks

Question 35

A. Prove that $(1-\sin x)(\sec x+\tan x)=\cos x$. 2

Solution	Comment(s)
$\begin{aligned} \text{LHS} &= (1-\sin x)(\sec x+\tan x) \\ &= (1-\sin x)\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right) \\ &= (1-\sin x)\left(\frac{1+\sin x}{\cos x}\right) \\ &= \frac{(1-\sin x)(1+\sin x)}{\cos x} \\ &= \frac{1-\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x \\ &= \text{RHS} \end{aligned}$	Alternatively, by expanding the LHS: $\begin{aligned} \text{LHS} &= \sec x+\tan x-\sin x\sec x-\sin x\tan x \\ &= \sec x+\tan x-\frac{\sin x}{\cos x}-\frac{\sin^2 x}{\cos x} \\ &= \frac{1}{\cos x}+\tan x-\tan x-\frac{\sin^2 x}{\cos x} \\ &= \frac{1-\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x \\ &= \text{RHS} \end{aligned}$ ~ Students generally performed well on this question.

B. Hence or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x(1-\sin x)(\sec x+\tan x)\,dx$. 3

Solution	Comment(s)
From Part A: $\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \sin^2 x\,(1-\sin x)(\sec x+\tan x)\,dx \\ &= \int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx \end{aligned}$ Using the reverse chain rule formula from the reference sheet: $\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1}+c$ Hence: $\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos x\sin^2 x\,dx \\ &= \left[\frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{3}\sin^3\frac{\pi}{2}-\frac{1}{3}\sin^3 0 \\ &= \frac{1}{3} \end{aligned}$	Alternatively for ME1 students, letting $u=\sin x$: $\begin{aligned} \frac{du}{dx} &= \cos x \\ dx &= \frac{du}{\cos x} \end{aligned}$ Hence: $\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos x\sin^2 x\,dx \\ &= \int_0^1 u^2\cos x\times\frac{1}{\cos x}\,du \\ &= \int_0^1 u^2\,du \\ &= \left[\frac{u^3}{3}\right]_0^1 \\ &= \frac{1}{3}-0 \\ &= \frac{1}{3} \end{aligned}$ ~ Most students could not make any progress beyond replacing $(1-\sin x)(\sec x+\tan x)$ with $\cos x$.

Question 36

Georgie has borrowed \$25 000 from a bank with interest accrued and repayments made monthly.

The following table lists the progress for the first 6 months of the loan.

Month Number	Opening Balance	Monthly Interest	Monthly Repayment	End Balance
1	\$25 000. 00	\$112. 50	\$400. 00	\$24 712. 50
2	\$24 712. 50	\$111. 21	\$400. 00	\$24 423. 71
3	\$24 423. 71	\$109. 91	\$400. 00	\$24 133. 62
4	\$24 133. 62	\$108. 60	\$400. 00	\$23 842. 22
5	\$23 842. 22	\$107. 29	\$400. 00	\$23 549. 51
6	\$23 549. 51	\$105. 97	\$400. 00	\$23 255. 48
7		A		B

A. What is the per annum rate of interest? 1

Solution	Comment(s)
Let R be the **monthly** interest rate. Also, let P_n and I_n be the opening balance and monthly interest on the nth month respectively. In the first month: $$\begin{aligned} I_1 &= P_1R \\ 112.50 &= 25\,000R \\ R &= \frac{112.50}{25\,000} \\ &= 0.004\,5 \end{aligned}$$ As R is the monthly rate, the annual rate is: $$0.004\,5 \times 12 = 0.054$$	Common error(s): • Giving the monthly rate instead of the annual rate. • Writing the answer as 0. 054% instead of 5. 4% or 0. 054.

B. Determine the values of A and B. 2

Solution	Comment(s)
Let A_n be the end balance on the nth month. Notice from the table that $P_n = A_{n-1}$. Hence: $$\begin{aligned} I_7 &= P_7R \\ &= A_6R \\ &= 23\,255.48 \times 0.004\,5 \\ &= 104.65 \end{aligned}$$ The monthly repayment is $\$400$, so: $$\begin{aligned} B &= P_7 + I_7 - 400 \\ &= 23\,255.48 + 104.65 - 400 \\ &= 22\,960.13 \end{aligned}$$	Common error(s): • Finding $P_7(1+R)$ instead of P_7R for the interest.

Question 36 (continued)

Month Number	Opening Balance	Monthly Interest	Monthly Repayment	End Balance
1	\$25 000. 00	\$112. 50	\$400. 00	\$24 712. 50
2	\$24 712. 50	\$111. 21	\$400. 00	\$24 423. 71
3	\$24 423. 71	\$109. 91	\$400. 00	\$24 133. 62
4	\$24 133. 62	\$108. 60	\$400. 00	\$23 842. 22
5	\$23 842. 22	\$107. 29	\$400. 00	\$23 549. 51
6	\$23 549. 51	\$105. 97	\$400. 00	\$23 255. 48
7		A		B

C. After how many months will Georgie have paid off the loan? 4

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>

Let $P = 25\ 000$ and $M = 400$.

Then:

$$\begin{aligned} A_1 &= P(1+R) - M \\ A_2 &= A_1(1+R) - M \\ &= P(1+R)^2 - M(1+R) - M \\ A_3 &= A_2(1+R) - M \\ &= P(1+R)^3 - M(1+R)^2 - M(1+R) - M \end{aligned}$$

Generalising to A_n:

$$\begin{aligned} A_n &= P(1+R)^n - M - M(1+R) - M(1+R)^2 \\ &\quad - \cdots - M(1+R)^{n-1} \\ &= P(1+R)^n - M(1 + (1+R) + (1+R)^2 \\ &\quad + \cdots + (1+R)^{n-1}) \end{aligned}$$

As $1 + (1+R) + (1+R)^2 + \cdots + (1+R)^{n-1}$ is a GP with n terms:

$$a = 1$$
$$r = 1 + R$$

Hence:

$$\begin{aligned} A_n &= P(1+R)^n - M\frac{(1+R)^n - 1}{(1+R) - 1} \\ &= P(1+R)^n - M\frac{(1+R)^n - 1}{R} \end{aligned}$$

Solution continues on next page.

</td><td>

A fair number of students were able to correctly deduce that 74 months were needed to pay off the loan, which was pleasing to see.

However, a large portion of responses either didn't give an explicit expression for A_3 or skipped straight to $A_n = 0$.

These responses were penalised accordingly.

Students should also note that using pronumerals instead of numbers could have saved them some time in writing their responses.

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</table>

Question 36 (continued)

Month Number	Opening Balance	Monthly Interest	Monthly Repayment	End Balance
1	\$25 000. 00	\$112. 50	\$400. 00	\$24 712. 50
2	\$24 712. 50	\$111. 21	\$400. 00	\$24 423. 71
3	\$24 423. 71	\$109. 91	\$400. 00	\$24 133. 62
4	\$24 133. 62	\$108. 60	\$400. 00	\$23 842. 22
5	\$23 842. 22	\$107. 29	\$400. 00	\$23 549. 51
6	\$23 549. 51	\$105. 97	\$400. 00	\$23 255. 48
7		A		B

C. After how many months will Georgie have paid off the loan? 4

Solution	Comment(s)
Loan is paid off at $A_n = 0$, so: $$0 = P(1+R)^n - M\frac{(1+R)^n - 1}{R}$$ $$P(1+R)^n = M\frac{(1+R)^n - 1}{R}$$ $$= \frac{M}{R}(1+R)^n - \frac{M}{R}$$ $$\frac{M}{R} = \left(\frac{M}{R} - P\right)(1+R)^n$$ $$(1+R)^n = \frac{M}{R\left(\frac{M}{R} - P\right)}$$ $$= \frac{M}{M - PR}$$ Taking the log of both sides with base $1+R$: $$n = \log_{1+R}\frac{M}{M-PR}$$ $$= \log_{1.004\,5}\frac{400}{400 - 25\,000 \times 0.004\,5}$$ $$= 73.6\ (1\ d.p.)$$ Hence, the loan is paid off on the 74th month.	

Question 37

A. If $f(x)=\dfrac{\ln x}{x}$, by differentiating, show that $f'(x) < 0$ for $x > e$. 2

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>

Using the quotient rule:

$$f'(x) = \frac{\left(x \times \frac{1}{x}\right) - (\ln x \times 1)}{x^2}$$
$$= \frac{1 - \ln x}{x^2}$$

Since $x > e$, $x \neq 0$.
Hence, subbing $f'(x) = 0$:

$$0 = \frac{1 - \ln x}{x^2}$$
$$\ln x = 1$$
$$x = e$$

Constructing a table of first derivatives:

x	2	e	3
$f'(x)$	0. 076	0	–0. 011

There are no other stationary points for $f(x)$.
Hence, $f'(x) < 0$ for $x > e$.

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Alternatively, given $x > e$:

$$\ln x > 1$$
$$1 < \ln x$$
$$1 - \ln x < 0$$

As $x^2 > 0$:

$$\frac{1 - \ln x}{x^2} < 0$$
$$f'(x) < 0$$

~

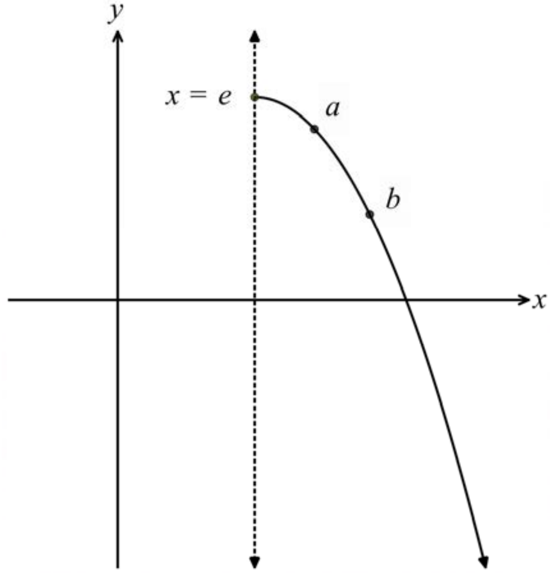
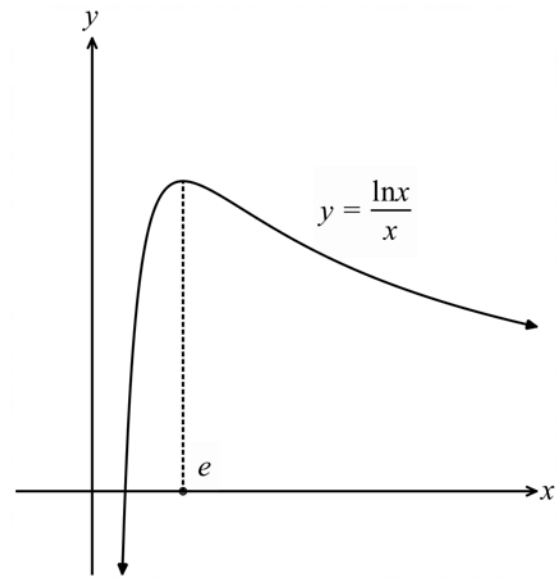
Students should note that the question asked them to show that $f'(x) < 0$, not that $x > e$.

That being said, a fair portion of students received full marks for this question, which was impressive given how late in the paper this question was.

</td></tr>
</table>

Question 37 (continued)

B. Hence, if $e < a < b$, show that $a^b > b^a$. 3

Solution	Comment(s)
From Part A, $f(x)$ is decreasing for $x > e$, as shown by the graph of some arbitrary function below:  Hence, $f(a) > f(b)$ for $e < a < b$, so: $\frac{\ln a}{a} > \frac{\ln b}{b}$ $b \ln a > a \ln b$ As $p \log x = \log(x^p)$: $\ln(a^b) > \ln(b^a)$ Taking the exponential on both sides with base e: $a^b > b^a$	The actual graph of $f(x) = \frac{\ln x}{x}$ is included below for reference:  Students should note that a proof question can't begin with the assumption that the expression to prove is true and can be manipulated freely. Responses that began with this assumption couldn't score any marks.

End of solutions

Section I Multiple Choice Solutions

1 Which of the following gives the domain of $y = \dfrac{1}{\sqrt{4-5x}}$?

A. $x > \dfrac{4}{5}$

B. $x < \dfrac{4}{5}$

C. $x \geq \dfrac{4}{5}$

D. $x \leq \dfrac{4}{5}$

A	9
B	165
C	2
D	9

Need $4 - 5x > 0$

2 The average mass of 6 bags of lollies is 0.4 kg.

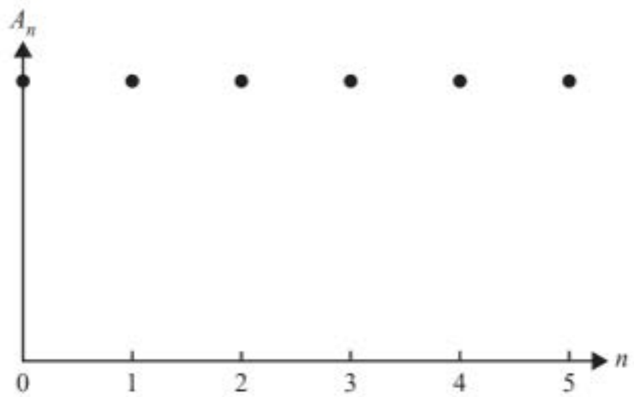
What could be the mass of the heaviest bag?

A. 0.3 kg B. 0.45 kg C. 2.5 kg D. 3 kg

Total mass = 6 × 0.4 = 2.4 kg
Options C and D are excluded.
Also, the heaviest bag should have a mass above the average which is 0.4 kg

A	3
B	174
C	6
D	2

3 The graph below represents the value A_n, in dollars, of an annuity investment for five time periods.



Which of the following recurrence relations could match this graphical representation?

A. $A_0 = 200\,000,\ A_{n+1} = 1.015A_n - 2500$

B. $A_0 = 200\,000,\ A_{n+1} = 1.025A_n - 5000$

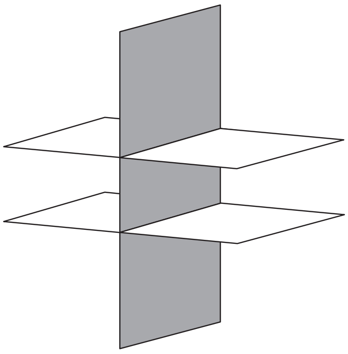
C. $A_0 = 200\,000,\ A_{n+1} = 1.035A_n - 5500$

D. $A_0 = 200\,000,\ A_{n+1} = 1.04A_n - 6000$

A	3
B	179
C	2
D	1

As 0.025 × 200 000 = 5000 then the value won't change.

4 The geometric interpretation of a certain system of three equations with no solution is shown below.



A	34
B	25
C	46
D	80

Given two of the equations are $x + y - z = 0.5$ and $x - y - z = 0.5$, which of the following could be the third equation?

A. $2x - 2y - 2z = 1$

B. $2x + 2y - 2z = 1$

C. $2x - 2y + 2z = 3$

D. $2x + 2y - 2z = 3$

Options A and B are the same as the above two equations.
With option D the equation is parallel but not equal to $x + y - z = 0.5$.

5 If $f(x) = e^{6-2x}$, what is the value of $f'(2)$?

A. e^2

B. $-e^2$

C. $2e^2$

D. $-2e^2$

A	2
B	0
C	5
D	178

$f'(x) = -2e^{6-2x}$

6 Let $f'(x) = g'(x) + 3$, $f(0) = 1$, and $g(0) = 1$.

Which of the following is correct?

A. $f(x) = g(x) + 1$

B. $f(x) = g(x) + 3$

C. $f(x) = g(x) + 3x$

D. $f(x) = g(x) + 3x + 1$

$f'(x) - g'(x) = 3 \Rightarrow f(x) - g(x) = 3x + C$
$\therefore f(0) - g(0) = 0 + C \Rightarrow C = 0$

A	1
B	4
C	159
D	21

7 The function $p(x) = x^3$ is an example of a polynomial with the property that there is a point of the graph $y = p(x)$ with zero derivative which is neither a local maximum nor a local minimum.

Which of the following polynomials has the same property?

A. $y = x^3 - 3x^2 + x$ B. $y = x^3 - 3x^2 + 2x$

C. $y = x^3 - 3x^2 + 3x$ D. $y = x^3 - 3x^2 + 4x$

A	24
B	19
C	125
D	17

Note that $y = x^3 - 3x^2 + 3x$ is equivalent to $y - 1 = x^3 - 3x^2 + 3x - 1$ i.e. $y = (x - 1)^3 + 1$

8 Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m, and n are positive integers.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following MUST be true?

A. $\frac{m}{n}$ is an integer. B. $\frac{n}{m}$ is an integer.

C. $\frac{a}{b}$ is an integer. D. $\frac{b}{a}$ is an integer.

A	40
B	26
C	33
D	86

$$amx^{m-1} = \frac{b}{n+1}x^{n+1}$$

So $b = am(n+1) \in \mathbb{Z}$

$$\therefore \frac{b}{a} = m(n+1) \in \mathbb{Z}$$

9 What is the maximum value of $\cos(2x + 30°)(\,1 - \cos(2x + 30°)\,)$?

A. $\frac{1}{4}$ B. $\frac{1}{2}$

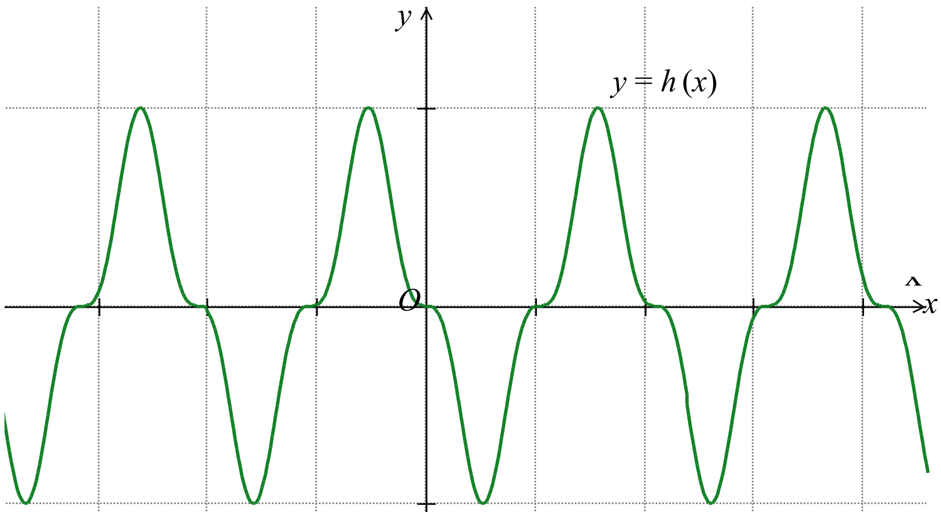
C. $\frac{3}{4}$ D. 1

A	133
B	18
C	7
D	27

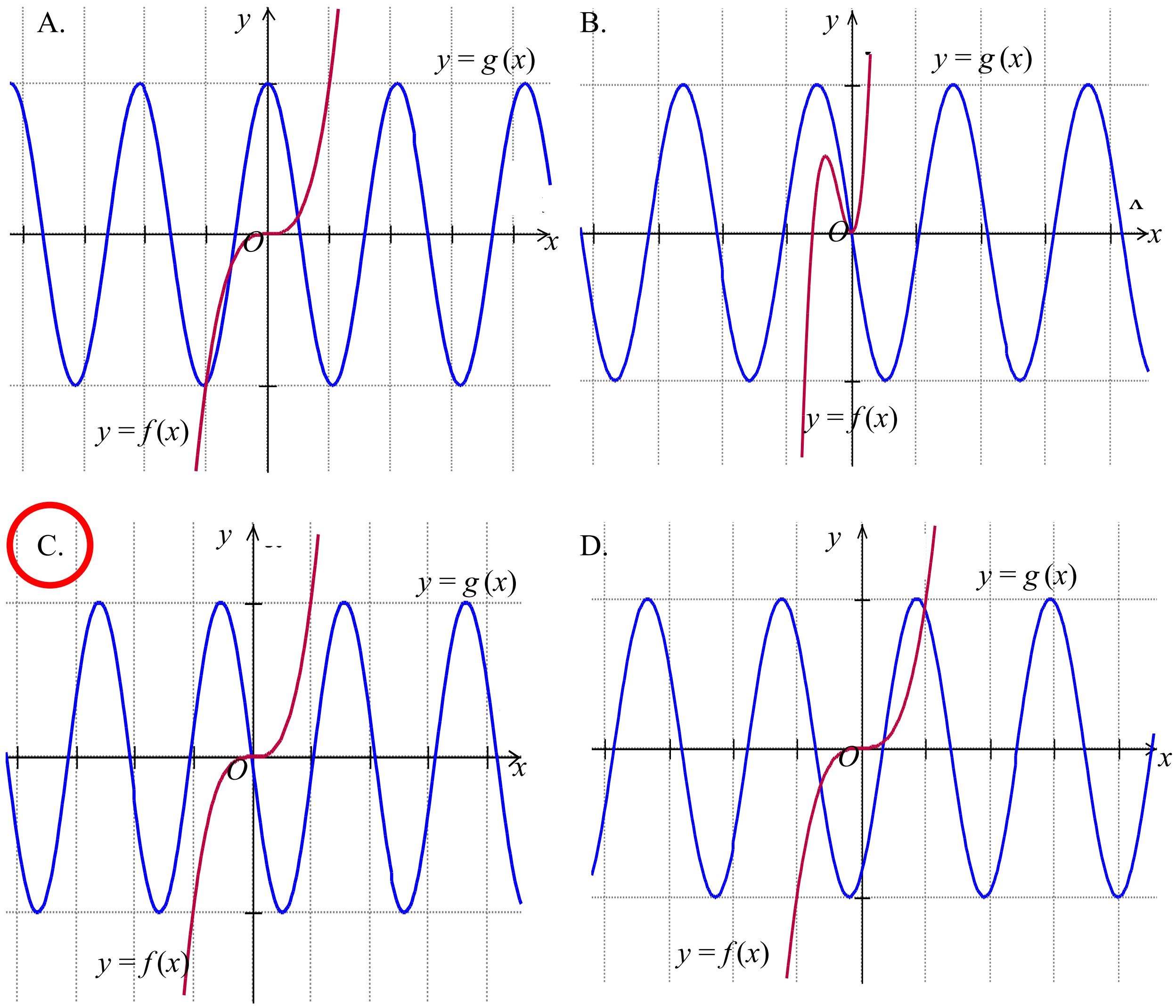
Let $u = \cos(2x + 30°)$
$\therefore \cos(2x + 30°)(\,1 - \cos(2x + 30°)\,) = u(1 - u)$

The maximum value of $u(1 - u)$ is $\frac{1}{4}$ at $u = \frac{1}{2}$

10 The diagram shows the graph of the composite function $h(x) = f(g(x))$.



Which of the following is the best option to represent the graphs of the two functions $f(x)$ and $g(x)$?



$h(x)$ is an odd function. If $g(x)$ is even then $h(x)$ is even. So A is out.
In B and D one of the functions is neither odd nor even.

A	10
B	17
C	146
D	12

End of Section I